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A FRACTURE CRITERION OF A PENNY-SHAPED CRACK IN TRANSVERSELY ISOTROPIC PIEZOELECTRIC MEDIA

JIN H. HUANG
Department of Mechanical Engineering, Feng Chia University, Taichung 407, Taiwan, Republic of China

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Abstract—By utilizing the eigenstrain formulation and Cauchy's residue theorem, a unified explicit expression for the electroelastic fields inside a flat ellipsoidal crack embedded in an infinite piezoelectric solid subjected to electromechanical loads is presented. In particular, an explicit expression is obtained for a penny-shaped crack in a transversely isotropic piezoelectric medium. Three loading cases, a simple tension, a pure shear, and an electric displacement, have been considered to examine the behavior of penny-shaped cracking. The results show that the applied shear stress does not couple with the electric displacement, unlike the simple tension case. Furthermore, the change of potential energy due to the presence of the crack is evaluated. With this result and based on the Griffith theory, the fracture stresses and critical electric displacement are presented in closed forms. Explicit expressions for stress and electric displacement intensity factors are also given. It is verified that the resulting fracture stresses and stress intensity factors can be reduced to those for uncoupled linear elastic fracture mechanics when piezoelectric coupling is absent and the material is isotropic. © 1997 Elsevier Science Ltd.

INTRODUCTION

As piezoelectric materials are extensively used as actuators, sensors, sonar projector and medical ultrasonic imaging applications, demand for advanced piezoelectric materials with high strength, high toughness, low thermal expansion coefficient, and low dielectric constant thus increases. It has become evident that conventional piezoelectric materials, such as lead zirconate titanate (PZT) and polyvinylidene difluoride (PVDF), can only partially meet the aforementioned requirements. In an effort to obtain a piezoelectric material with these competing properties, considerable research has been directed toward the development of piezoelectric composite materials engineered to incorporate inclusions. Combining two or more distinct constituents, piezoelectric composite materials can take the advantages of each constituent. However, the introduction of inclusions into base media will generally lead to the material being anisotropic and complicate. In some situations, it is a common occurrence for thin fracture cracks to be present in material body. Cracks may be such that they are detrimental (or possible beneficial) to the performance of the piezoelectric composites. Therefore, in order that the reliable service lifetime prediction of the composite can be obtained, it is necessary to clearly examine the electroelastic responses from a micromechanics point of view so that the influence of cracks can be understood thoroughly. Thus, the present paper is an attempt to fill this information need.

The study of piezoelectric crack problems is receiving increased attention. The most significant works in this area are those of Parton (1976), Deeg (1980), Sosa and Pak (1990), Sosa (1991), Pak (1992), Suo *et al.* (1992), and Wang (1994). More recently, Dunn (1994) applied the electroelastic Eshelby tensors (Dunn and Taya, 1992) to investigate the effects of crack face boundary conditions on the fracture mechanics of piezoelectric solids. Another rigorous fracture mechanics framework for a flat ellipsoidal flaw in a transversely isotropic piezoelectric solid was forwarded by Wang (1992, 1994). Through the eigenstrain formulation of Mura (1987) incorporating the equivalent inclusion method of Eshelby (1957), Wang (1994) obtained integral expressions to represent the crack opening displacement and the stress intensity factors. These solutions, however, are not explicit because the residues of the surface integral π and the equivalent eigenstrain that are the key ingredients in solving the crack problems have not been obtained in his papers. Consequently, his approach is limited in applications.

Based on the eigenstrain formulation, this paper is to develop a unified and simple explicit solution to the problems of a three-dimensional piezoelectric solid containing a flat ellipsoidal inclusion. The unified expression for the coupled elastic and electric fields in a flat ellipsoidal inclusion is written in the form of surface integral first, and reduced to a line integral by employing Cauchy's residue theorem. Then, the equivalent inclusion method is applied to obtain the closed-form expression for the fictitious eigenfield in a crack. Detail calculations are given when the crack is penny-shaped and the material is transversely isotropic subjected to constant applied mechanical loads and electric displacements. Finally, based on the Griffith (1921) theory, the critical states as well as the stress and electric displacement intensity factors of the crack are derived.

BASIC EQUATIONS

Consider an infinitely extended piezoelectric solid D containing an ellipsoidal piezoelectric inclusion Ω whose electroelastic moduli $\hat{C}_{iJMn}$ are the same as the matrix. Let Z^*_{Mn} be eigenfield in Ω, and zero in the matrix $D-\Omega$. Here the lowercase subscripts range from 1 to 3, while the uppercase subscripts range from 1 to 4. The electroelastic moduli $\hat{C}_{iJMn}$ and eigenfield Z^*_{Mn} are defined as follows:

$$\hat{C}_{iJMn} = \begin{cases} C_{ijmn} & J, M \leqslant 3, \\ e_{nij} & J \leqslant 3; M = 4, \\ e_{imn} & J = 4; M \leqslant 3, \\ -\kappa_{in} & J, M = 4, \end{cases} \qquad Z^*_{Mn} = \begin{cases} \varepsilon^*_{mn} & M \leqslant 3, \\ -E^*_n = \phi^*_{,n} & M = 4, \end{cases} \tag{1}$$

where C_{ijmn} is the elastic moduli measured at a constant electric field, e_{imn} is the piezoelectric coefficient measured at a constant strain or electric field, κ_{in} is the dielectric constant measured at a constant strain, ε^*_{mn} is the eigenstrain (or stress-free strain), E^*_n is the eigenelectric field (or electric displacement-free electric field), and ϕ^* is the eigenelectric potential.

The stress and electric displacement, Σ_{ij}, in the inclusion caused by Z^*_{Mn} uniformly distributed in Ω can be expressed as:

$$\sum_{iJ} = \begin{cases} \sigma_{ij} = C_{ijmn}(u_{m,n} - \varepsilon^*_{mn}) - e_{nij}(E_n - E^*_n), & J = \leqslant 3, \\ D_i = e_{imn}(u_{m,n} - \varepsilon^*_{mn}) + \kappa_{in}(E_n - E^*_n), & J = 4, \end{cases} \tag{2}$$

where u_m, ε_{mn}, σ_{ij}, ϕ, E_n and D_i are the elastic displacement, strain, stress electric potential, electric field, electric displacement, respectively. In the absence of body forces and free electric charges in the piezoelectric material, the equation of elastic equilibrium and Gauss' law of electrostatics are expressed as

$$\sum_{iJ,i} = \begin{cases} \sigma_{ij,i} = 0 & J \leqslant 3, \\ D_{i,i} = 0 & J = 4. \end{cases} \tag{3}$$

Substitution of eqn (3) into (2) leads to

$$\hat{C}_{iJMn} U_{M,ni} = \hat{C}_{iJMn} Z^*_{Mn,i}, \tag{4}$$

where

$$U_{M,n} = \begin{cases} u_{m,n} & M \leqslant 3, \\ \phi_{,n} = -E_n & M = 4. \end{cases} \tag{5}$$

The fundamental eqn (4) can be solved for $U_{M,n}$ by using the standard Fourier transform technique to give (Huang and Yu, 1994)

$$U_{A,b} = \frac{a_1 a_2 a_3}{4\pi} \hat{C}_{qRMn} Z^*_{Mn} \int_{S^2} G_{ARqb}(\bar{\xi}) \zeta^{-3} \, \mathrm{d}S(\bar{\xi}), \tag{6}$$

where a_1, a_2, and a_3 are the lengths of the semiaxes of the ellipsoid, S^2 is the unit sphere $\bar{\xi}_i \bar{\xi}_i = 1$, and

$$G_{ARqb}(\bar{\xi}) = \bar{\xi}_q \bar{\xi}_b N_{AR}(\bar{\xi}) / D(\bar{\xi}), \tag{7}$$

with $N_{AR}(\bar{\xi})$ and $D(\bar{\xi})$ being the cofactor and the determinant of the 4×4 matrix $\hat{C}_{iARn} \bar{\xi}_i \bar{\xi}_n$, respectively.

FLAT ELLIPSOIDAL INCLUSION

When the inclusion is a flat ellipsoid ($a_3 \ll a_1, a_2$), eqn (6) can be further simplified by the following transformation (Mura, 1987):

$$\bar{\xi}_1 = (1-\bar{\xi}_3^2)^{1/2} \cos\theta, \quad \bar{\xi}_2 = (1-\bar{\xi}_3^2)^{1/2} \sin\theta, \quad t = \bar{\xi}_3/(1-\bar{\xi}_3^2)^{1/2},$$
$$\zeta = (\zeta_1^2+\zeta_2^2+\zeta_3^2)^{1/2} = (a_1^2\bar{\xi}_1^2 + a_2^2\bar{\xi}_2^2 + a_3^2\bar{\xi}_3^2)^{1/2}, \quad \mathrm{d}S(\bar{\xi}) = \mathrm{d}\bar{\xi}_3 \, \mathrm{d}\theta = (1-\bar{\xi}_3^2)^{3/2} \, \mathrm{d}t \, \mathrm{d}\theta. \tag{8}$$

Since $G_{ARqb}(\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3)$ is a homogeneous function of degree zero, it can be expressed as

$$G_{ARqb}[(1-\bar{\xi}_3^2)^{1/2} \cos\theta, (1-\bar{\xi}_3^2)^{1/2} \sin\theta, \bar{\xi}_3] = G_{ARqb}(\cos\theta, \sin\theta, t). \tag{9}$$

Substituting eqns (8) and (9) into (6) results in

$$U_{A,b} = \frac{1}{4\pi} \hat{C}_{qRMn} Z^*_{Mn} Y_{ARqb}, \tag{10}$$

where

$$Y_{ARqb} = a_1 a_2 a_3 \int_0^{2\pi} \mathrm{d}\theta \int_{-\infty}^{\infty} \frac{G_{ARqb}(\cos\theta, \sin\theta, t)}{(a_1^2 \cos^2\theta + a_2^2 \sin^2\theta + a_3^2 t^2)^{3/2}} \mathrm{d}t. \tag{11}$$

With use of the identity

$$\frac{\partial}{\partial t}\left[\frac{t}{(a_1^2 \cos^2\theta + a_2^2 \sin^2\theta + a_3^2 t^2)^{1/2}}\right] = \frac{a_1^2 \cos^2\theta + a_2^2 \sin^2\theta}{(a_1^2 \cos^2\theta + a_2^2 \sin^2\theta + a_3^2 t^2)^{3/2}}, \tag{12}$$

eqn (11) is written as

$$Y_{ARqb} = a_1 a_2 a_3 \int_0^{2\pi} \frac{\mathrm{d}\theta}{a_1^2 \cos^2\theta + a_2^2 \sin^2\theta} \int_{-\infty}^{\infty} G_{ARqb}(\cos\theta, \sin\theta, t) \cdot \frac{\partial}{\partial t} \frac{t}{(a_1^2 \cos^2\theta + a_2^2 \sin^2\theta + a_3^2 t^2)^{1/2}} \mathrm{d}t. \tag{13}$$

When integrating by parts with respect to t and using the property $G_{ARqb}(\cos\theta, \sin\theta, t) = G_{ARqb}(\cos\theta/t, \sin\theta/t, 1) = G_{ARqb}(0, 0, 1)$ as $t \to \pm\infty$, eqn (13) can be shown to be

$$Y_{ARqb} = 2a_1 a_2 G_{ARqb}(0,0,1) \int_0^{2\pi} \frac{d\theta}{a_1^2\cos^2\theta - a_2^2\sin^2\theta} - a_1 a_2 a_3 \int_0^{2\pi} \frac{d\theta}{a_1^2\cos^2\theta + a_2^2\sin^2\theta} \cdot \int_{-\infty}^{\infty} \frac{t}{(a_1^2\cos^2\theta + a_2^2\sin^2\theta + a_3^2 t^2)^{1/2}} \frac{\partial}{\partial t} G_{ARqb}(\cos\theta, \sin\theta, t)\, dt. \quad (14)$$

Since the first term on the right hand of eqn (14) can be obtained as

$$2a_1 a_2 a_3 G_{ARqb}(0,0,1) \int_0^{2\pi} \frac{d\theta}{a_1^2\cos^2\theta + a_2^2\sin^2\theta} = 4\pi G_{ARqb}(0,0,1), \quad (15)$$

eqn (14) is expressed as

$$Y_{ARqb} = 4\pi G_{ARqb}(0,0,1) - a_3 \Pi_{ARqb}, \quad (16)$$

where

$$\Pi_{ARqb} = a_1 a_2 \int_0^{2\pi} \int_{-1}^{1} \frac{\xi_3/(1-\xi_3^2)^{1/2}}{(a_1^2\cos^2\theta + a_2^2\sin^2\theta)^{3/2}} \frac{\partial}{\partial \xi_3} [G_{ARqb}(\xi_1, \xi_2, \xi_3)]\, d\xi_3\, d\theta. \quad (17)$$

Substituting eqns (16) and (17) into (6), we finally have

$$U_{A,b} = \frac{1}{4\pi} \hat{C}_{qRMn} Z^*_{Mn} [4\pi G_{ARqb}(0,0,1) - a_3 \Pi_{ARqb}]. \quad (18)$$

It is seen from the above equation that the evaluation of Π_{ARqb} is of primary importance as with its analytical expression in hand, the electroelastic fields given in eqns (2) and (18) can easily be solved. Therefore, it will be devoted to explore analytical expressions for Π_{ARqb} in the next section.

EVALUATION OF Π INTEGRAL

In general, eqn (17) cannot be expressed analytically. However, it can be expressed in a complex variable form so that the eqn (17) in terms of complex variable z, while $\cos\theta$ and $d\theta$ are changed to

$$\cos\theta = (z + z^{-1})/2, \quad \sin\theta = (z - z^{-1})/2i, \quad d\theta = dz/iz. \quad (19)$$

Then, using Cauchy's residue theorem, eqn (17) can be carried out explicitly as

$$\Pi_{ARqb} = 2\pi \int_{-1}^{1} R[f]\, d\xi_3, \quad (20)$$

where

$$R[f] = a_1 a_2 \int_0^{2\pi} \frac{\xi_3/(1-\xi_3^2)^{1/2}}{(a_1^2\cos^2\theta + a_2^2\sin^2\theta)^{3/2}} [G_{ARqb}(\xi_1, \xi_2, \xi_3)]\, d\theta, \quad (21)$$

which denotes the sum of the residues of the complex function f existing within the unit circle $|z| = 1$ under a fixed value of $\bar{\xi}_3$. It is noted that there are twelve poles in eqn (21).

Next, Π_{ARqb} for a penny-shaped inclusion ($a_3 \ll a_1 = a_2 = a$) which is a special shape of a flat ellipsoid will be explored. Suppose that the crystalline directions of the piezoelectric material are coincident with the principal axes of the inclusion, and both the piezoelectric matrix and piezoelectric inclusion are considered to be transversely isotropic with x_3 as a symmetry axis. Upon substituting $D(\bar{\xi})$ and $N_{AR}(\bar{\xi})$ into eqn (21), and evaluating the residues, all other components of Π_{ARqb} are zero except the following:

$$\Pi_{1111} = \Pi_{2222} = \frac{-\pi}{2a}\int_0^1 A^{-1}(1-x^2)^{-1/2}[(3a_{11}+b_{11})(1-x^2)^3+(4d_{11}+3f_{11}+h_{11})$$
$$\cdot(1-x^2)^2x^2+(4m_{11}+3p_{11}+q_{11})(1-x^2)x^4+4r_{11}x^6]\,\mathrm{d}x,$$

$$\Pi_{1122} = \Pi_{2211} = \frac{-\pi}{2a}\int_0^1 A^{-1}(1-x^2)^{-1/2}[(a_{11}+3b_{11})(1-x^2)^3+(4d_{11}+f_{11}+3h_{11})$$
$$\cdot(1-x^2)^2x^2+(4m_{11}+p_{11}+3q_{11})(1-x^2)x^4+4r_{11}x^6]\,\mathrm{d}x,$$

$$\Pi_{1133} = \Pi_{2233} = \frac{-2\pi}{a}\int_0^1 A^{-2}x^2(1-x^2)^{-1/2}[\alpha_0(1-x^2)^6+\alpha_1(1-x^2)^5x^2$$
$$+\alpha_3(1-x^2)^3x^6+\alpha_4(1-x^2)^2x^8+\alpha_5(1-x^2)^2x^{10}+\alpha_6x^{12}]\,\mathrm{d}x,$$

$$\Pi_{1212} = \Pi_{1221} = \Pi_{2112} = \Pi_{2121}$$
$$= \frac{-\pi}{2a}\int_0^1 A^{-1}(1-x^2)^{1/2}[a_{12}(1-x^2)^2+b_{12}(1-x^2)x^2+d_{12}x^4]\,\mathrm{d}x,$$

$$\Pi_{1313} = \Pi_{1331} = \Pi_{3113} = \Pi_{3131} = \Pi_{2323} = \Pi_{2332} = \Pi_{3223} = \Pi_{3232}$$
$$= \frac{-2\pi}{a}\int_0^1 B^{-1}(1-x^2)^{-1/2}x^2[d_{13}(1-x^2)+f_{13}x^2]\,\mathrm{d}x,$$

$$\Pi_{1413} = \Pi_{1431} = \Pi_{4113} = \Pi_{4131} = \Pi_{2423} = \Pi_{2432} = \Pi_{4223} = \Pi_{4232}$$
$$= \frac{-2\pi}{a}\int_0^1 B^{-1}(1-x^2)^{-1/2}x^2[d_{14}(1-x^2)+f_{14}x^2]\,\mathrm{d}x,$$

$$\Pi_{3311} = \Pi_{3322} = \frac{-2\pi}{a}\int_0^1 B^{-1}(1-x^2)^{-1/2}[d_{33}(1-x^2)^2+f_{33}(1-x^2)x^2+h_{33}x^4]\,\mathrm{d}x,$$

$$\Pi_{3333} = \frac{8\pi}{a}\int_0^1 B^{-2}x^2(1-x^2)^{-1/2}[(h_0h_{33}-m_0f_{33})x^8+2(f_0h_{33}-m_0d_{33})(1-x^2)x^6$$
$$+(f_0f_{33}-h_0d_{33}+3d_0h_{33})(1-x^2)^2x^4+2d_0f_{33}(1-x^2)^3x^2$$
$$+d_0d_{33}(1-x^2)^4]\,\mathrm{d}x,$$

$$\Pi_{3411} = \Pi_{4311} = \Pi_{3422} = \Pi_{4322}$$
$$= \frac{-2\pi}{a}\int_0^1 B^{-1}(1-x^2)^{-1/2}[d_{34}(1-x^2)^2+f_{34}(1-x^2)x^2+h_{34}x^4]\,\mathrm{d}x,$$

$$\Pi_{3433} = \Pi_{4333} = \frac{8\pi}{a}\int_0^1 B^{-2}x^2(1-x^2)^{-1/2}[(h_0h_{34}-m_0f_{34})x^8+2(f_0h_{34}-m_0d_{34})$$
$$\cdot(1-x^2)x^6+(f_0f_{34}-h_0d_{34}+3d_0h_{34})(1-x^2)^2x^4+2d_0f_{34}(1-x^2)^3x^2$$
$$+d_0d_{34}(1-x^2)^4]\,\mathrm{d}x,$$

$$\Pi_{4411} = \Pi_{4422} = \frac{-2\pi}{a}\int_0^1 B^{-1}(1-x^2)^{-1/2}[d_{44}(1-x^2)^2+f_{44}(1-x^2)x^2+h_{44}x^4]\,dx,$$

$$\Pi_{4433} = \frac{8\pi}{a}\int_0^1 B^{-2}x^2(1-x^2)^{-1/2}[(h_0h_{44}-m_0f_{44})x^8+2(f_0h_{44}-m_0d_{44})$$

$$\cdot(1-x^2)x^6+(f_0f_{44}-h_0d_{44}+3d_0h_{44})(1-x^2)^2x^4+2d_0f_{44}(1-x^2)^3x^2$$

$$+d_0d_{44}(1-x^2)^4]\,dx, \tag{22}$$

where

$$A = [a_0(1-x^2)+b_0x^2]B,$$

$$B = d_0(1-x^2)^3+f_0(1-x^2)^2x^2+h_0(1-x^2)x^4+m_0x^6. \tag{23}$$

The coefficients $a_0 \sim m_0$ and $a_{IJ} \sim r_{IJ}$ in eqn (22) have been given in the paper of Huang and Yu (1994), and $\alpha_1 \sim \alpha_6$ in Π_{1133} are listed in the Appendix. It is noted that when piezoelectric coupling is absent, i.e., $e_{imn} = 0$, the expressions in eqn (2) agree with Mura's (1987) results for a transversely isotropic elastic medium. However, Π_{1122} and Π_{3311} on page 258 of Mura's book are incorrect.

EQUIVALENT INCLUSION METHOD

In this section, the equivalent inclusion idea of Eshelby (1957) is applied to solve the piezoelectric crack problems. An analytical method for determining the closed-form solutions of the equivalent eigenfields is presented.

Consider a sufficiently large piezoelectric solid with electroelastic moduli $\hat{C}_{iJAb}$, containing an ellipsoidal inhomogeneity Ω with electroelastic moduli $\hat{C}^*_{iJAb}$. When the piezoelectric material is subjected to the uniform applied stress and electric displacement Σ^0_{iJ}, the stress and electric displacement will be $\Sigma^0_{iJ}+\Sigma_{iJ}$ due to the presence of Ω. Here Σ_{iJ} represents the stress and electric displacement disturbance caused by the presence of the inhomogeneity. According to the equivalent inclusion method (Eshelby, 1957), the stress and electric displacement in Ω can be simulated by those in an equivalent inclusion with the electroelastic constants of the matrix and a uniform fictitious eigenfield, Z^*_{Mn}, namely,

$$\sum_{iJ}^{0}+\sum_{iJ} = \hat{C}^*_{iJAb}(U^0_{A,b}+U_{A,b})$$

$$= \hat{C}_{iJAb}(U^0_{A,b}+U_{A,b}-Z^*_{Ab}) \quad \text{in } \Omega. \tag{24}$$

Since a crack can be understood as the inhomogeneity where its electroelastic moduli vanish (by setting $C^*_{ijmn} = 0$, and $e^*_{imn} = 0$, and $\kappa^*_{in} \approx 0$), the equivalency condition (24) becomes

$$\sum_{iJ}^{0}+\sum_{iJ} = \hat{C}_{iJAb}(U^0_{A,b}+U_{A,b}-Z^*_{Ab}) = 0 \quad \text{in } \Omega. \tag{25}$$

Substituting eqn (18) into the foregoing equation, the stress and electric displacement disturbances caused by the crack are expressed as

$$\sum_{iJ} = \hat{C}_{iJAb}(U_{A,b} - Z^*_{Ab})$$

$$= (K_{iJAb} - a_3 L_{iJAb}) Z^*_{Ab}, \tag{26}$$

where

$$K_{iJAb} = \hat{C}_{iJMn}\hat{C}_{aRAb}G_{MRqn}(0,0,1) - \hat{C}_{iJAb} \tag{27}$$

$$L_{iJAb} = \frac{1}{4\pi}\hat{C}_{iJMn}\hat{C}_{qRAb}\Pi_{MRqn}. \tag{28}$$

It can easily be shown that K_{iJAb} vanishes if one of subscripts is three, i.e., the only nonzero components of K_{iJAb} are $K_{1111}(=K_{2222})$, $K_{1122}(=K_{2211})$, K_{1212}, $K_{1441}(=K_{2442})$. In addition, since L_{iJAb} is independent of a_3, we obtain the following properties:

$$\lim_{a_3\to 0}(K_{iJAb} - a_3 L_{iJAb}) = K_{iJAb} \tag{29}$$

if none of the i, J, A, b equals 3, and

$$\lim_{a_3\to 0}(K_{iJAb} - a_3 L_{iJAb}) = -a_3 L_{iJAb} \tag{30}$$

otherwise. Using the results (26)–(30), eqn (25) when written out in detail gives a system of nine equations:

$$-\overset{0}{\sum_{11}} = K_{1111}Z^*_{11} + K_{1122}Z^*_{22} - L_{1133}(a_3Z^*_{33}) - 2L_{1123}(a_3Z^*_{23}) - 2L_{1113}(a_3Z^*_{13})$$
$$+ 2K_{1112}Z^*_{12} + K_{1141}Z^*_{41} + K_{1142}Z^*_{42} - L_{1143}(a_3Z^*_{43}), \tag{31}$$

$$-\overset{0}{\sum_{22}} = K_{2211}Z^*_{11} + K_{2222}Z^*_{22} - L_{2233}(a_3Z^*_{33}) - 2L_{2223}(a_3Z^*_{23}) - 2L_{2213}(a_3Z^*_{13})$$
$$+ 2K_{2212}Z^*_{12} + K_{2241}Z^*_{41} + K_{2242}Z^*_{42} - L_{2243}(a_3Z^*_{43}), \tag{32}$$

$$-\overset{0}{\sum_{33}} = -L_{3311}(a_3Z^*_{11}) - L_{3322}(a_3Z^*_{22}) - L_{3333}(a_3Z^*_{33}) - 2L_{3323}(a_3Z^*_{23}) - 2L_{3313}(a_3Z^*_{13})$$
$$- 2L_{3312}(a_3Z^*_{12}) - L_{3341}(a_3Z^*_{41}) - L_{3342}(a_3Z^*_{42}) - L_{3343}(a_3Z^*_{43}), \tag{33}$$

$$-\overset{0}{\sum_{23}} = -L_{2311}(a_3Z^*_{11}) - L_{2322}(a_3Z^*_{22}) - L_{2333}(a_3Z^*_{33}) - 2L_{2323}(a_3Z^*_{23}) - 2L_{2313}(a_3Z^*_{13})$$
$$- 2L_{2312}(a_3Z^*_{12}) - L_{2341}(a_3Z^*_{41}) - L_{2342}(a_3Z^*_{42}) - L_{2343}(a_3Z^*_{43}), \tag{34}$$

$$-\overset{0}{\sum_{13}} = -L_{1311}(a_3Z^*_{11}) - L_{1322}(a_3Z^*_{22}) - L_{1333}(a_3Z^*_{33}) - 2L_{1323}(a_3Z^*_{23}) - 2L_{1313}(a_3Z^*_{13})$$
$$- 2L_{1312}(a_3Z^*_{12}) - L_{1341}(a_3Z^*_{41}) - L_{1342}(a_3Z^*_{42}) - L_{1343}(a_3Z^*_{43}), \tag{35}$$

$$-\overset{0}{\sum_{12}} = K_{1211}Z^*_{11} + K_{1222}Z^*_{22} - L_{1233}(a_3Z^*_{33}) - 2L_{1223}(a_3Z^*_{23}) - 2L_{1213}(a_3Z^*_{13})$$
$$+ 2K_{1212}Z^*_{12} + K_{1241}Z^*_{41} + K_{1242}Z^*_{42} - L_{1243}(a_3Z^*_{43}), \tag{36}$$

$$-\Sigma^0_{14} = K_{1411}Z^*_{11} + K_{1422}Z^*_{22} - L_{1433}(a_3 Z^*_{33}) - 2L_{1423}(a_3 Z^*_{23}) - 2L_{1413}(a_3 Z^*_{13})$$

$$+2K_{1412}Z^*_{12} + K_{1441}Z^*_{41} + K_{1442}Z^*_{42} - L_{1443}(a_3 Z^*_{43}), \quad (37)$$

$$-\Sigma^0_{24} = K_{2411}Z^*_{11} + K_{2422}Z^*_{22} - L_{2433}(a_3 Z^*_{33}) - 2L_{2423}(a_3 Z^*_{23}) - 2L_{2413}(a_3 Z^*_{13})$$

$$+2K_{2412}Z^*_{12} + K_{2441}Z^*_{41} + K_{2442}Z^*_{42} - L_{2443}(a_3 Z^*_{43}), \quad (38)$$

$$-\Sigma^0_{34} = -L_{3411}(a_3 Z^*_{11}) - L_{3422}(a_3 Z^*_{22}) - L_{3433}(a_3 Z^*_{33}) - 2L_{3423}(a_3 Z^*_{23}) - 2L_{3413}(a_3 Z^*_{13})$$

$$-2L_{3412}(a_3 Z^*_{12}) - L_{3441}(a_3 Z^*_{41}) - L_{3442}(a_3 Z^*_{42}) - L_{3443}(a_3 Z^*_{43}). \quad (39)$$

Thus, the equivalent eigenstrain and eigenelectric field, Z^*_{Ab}, can be solved from this set of equations for a given Σ^0_{iJ}. For example, we consider the case where the acting plane of the applied stress and electric displacement Σ^0_{iJ} is parallel to the crack surfaces, namely, $\Sigma^0_{iJ} = \Sigma^0_{3J}$. From eqns (31), (32), (36), (37) and (38), we require that Z^*_{11}, Z^*_{22}, Z^*_{12}, Z^*_{41}, and Z^*_{42} remain finite values as a_3 goes to zero, and therefore

$$a_3 Z^*_{11} = a_3 Z^*_{22} = a_3 Z^*_{12} = a_3 Z^*_{41} = a_3 Z^*_{42} = 0. \quad (40)$$

Hence, eqns (33), (34), (35) and (39) respectively become

$$\Sigma^0_{33} = L_{3333}(a_3 Z^*_{33}) + 2L_{3323}(a_3 Z^*_{23}) + 2L_{3313}(a_3 Z^*_{13}) + L_{3343}(a_3 Z^*_{43}), \quad (41)$$

$$\Sigma^0_{23} = L_{2333}(a_3 Z^*_{33}) + 2L_{2323}(a_3 Z^*_{23}) + 2L_{2313}(a_3 Z^*_{13}) + L_{2343}(a_3 Z^*_{43}), \quad (42)$$

$$\Sigma^0_{13} = L_{1333}(a_3 Z^*_{33}) + 2L_{1323}(a_3 Z^*_{23}) + 2L_{1313}(a_3 Z^*_{13}) + L_{1343}(a_3 Z^*_{43}), \quad (43)$$

$$\Sigma^0_{34} = L_{3433}(a_3 Z^*_{33}) + 2L_{3423}(a_3 Z^*_{23}) + 2L_{3413}(a_3 Z^*_{13}) + L_{3443}(a_3 Z^*_{43}). \quad (44)$$

Furthermore, from eqn (28), the components of L_{iJAb} can be written in the following matrix form for a transversely isotropic piezoelectric material:

$$\begin{bmatrix} L_{1111} & L_{1122} & L_{1133} & 0 & 0 & 0 & 0 & 0 & L_{1143} \\ L_{1122} & L_{1111} & L_{1133} & 0 & 0 & 0 & 0 & 0 & L_{1143} \\ L_{3311} & L_{3311} & L_{3333} & 0 & 0 & 0 & 0 & 0 & L_{3343} \\ 0 & 0 & 0 & L_{2323} & 0 & 0 & 0 & L_{2342} & 0 \\ 0 & 0 & 0 & 0 & L_{1313} & 0 & L_{2342} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & L_{1212} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & L_{1413} & 0 & L_{1441} & 0 & 0 \\ 0 & 0 & 0 & L_{1413} & 0 & 0 & 0 & L_{1441} & 0 \\ L_{3411} & L_{3411} & L_{3433} & 0 & 0 & 0 & 0 & 0 & L_{3443} \end{bmatrix}. \quad (45)$$

Using the nonzero components in eqn (45), eqns (41)–(44) can be further reduced to

$$\sigma_{33}^{0} = L_{3333}(a_3 Z_{33}^{*}) + L_{3343}(a_3 Z_{43}^{*}), \tag{46}$$

$$\sigma_{23}^{0} = 2L_{2323}(a_3 Z_{23}^{*}), \tag{47}$$

$$\sigma_{13}^{0} = 2L_{1313}(a_3 Z_{13}^{*}), \tag{48}$$

$$D_{3}^{0} = L_{3433}(a_3 Z_{33}^{*}) + L_{3443}(a_3 Z_{43}^{*}). \tag{49}$$

Then, the equivalent eigenstrain and eigenelectric field are easily solved from these four equations as

$$a_3 Z_{33}^{*} = \frac{\sigma_{33}^{0} L_{3443} - D_{3}^{0} L_{3343}}{L_{3333} L_{3443} - L_{3343} L_{3433}}, \tag{50}$$

$$a_3 Z_{43}^{*} = \frac{D_{3}^{0} L_{3333} - \sigma_{33}^{0} L_{3433}}{L_{3333} L_{3443} - L_{3343} L_{3433}}, \tag{51}$$

$$a_3 Z_{13}^{*} = \frac{\sigma_{31}^{0}}{2L_{1313}}, \tag{52}$$

$$a_3 Z_{23}^{*} = \frac{\sigma_{23}^{0}}{2L_{2323}}. \tag{53}$$

By inspection of eqns (50)–(53) it is seen that the applied shear stress does not couple with the applied electric displacement, unlike the simple tension case.

CRITICAL STRESSES AND ELECTRICAL DISPLACEMENT

An important quantity in fracture mechanics is the interaction energy, ΔW, which is defined as the change of total potential energy due to the presence of the crack. When a piezoelectric material D contains a penny-shaped crack Ω and is subjected to the far-field traction and electric displacement, $\Sigma_{iJ}^{0} n_i$, on the boundary $|D|$ with outward unit normal vector n_i, the interaction energy is written as

$$\Delta W = \frac{1}{2}\int_{D}\left(\overset{0}{\sum_{iJ}} + \sum_{iJ}\right)(U_{J,i}^{0} + U_{J,i})\,\mathbf{dx} - \int_{|D|}\left(\overset{0}{\sum_{iJ}} n_i\right)(U_J^{0} + U_J)\,\mathbf{dx}$$
$$-\left[\frac{1}{2}\int_{D}\overset{0}{\sum_{iJ}} U_{J,i}^{0}\,\mathbf{dx} - \int_{|D|}\left(\overset{0}{\sum_{iJ}} n_i\right) U_J^{0}\,\mathbf{dx}\right]. \tag{54}$$

Applying Gauss' theorem and eqn (25) to the last equation yields

$$\Delta W = -\frac{1}{2}\int_{\Omega}\overset{0}{\sum_{iJ}} Z_{Ji}^{*}\,\mathbf{dx} = -\frac{2}{3}\pi a^2 a_3 \overset{0}{\sum_{iJ}} Z_{Ji}^{*}. \tag{55}$$

Next, three special cases of simple tension ($\Sigma_{33}^{0} = \sigma_{33}^{0}$), normal electrical displacement ($\Sigma_{34}^{0} = D_{3}^{0}$), and pure shear ($\Sigma_{31}^{0} = \sigma_{31}^{0}$) are considered separately. When the piezoelectric material is considered to be transversely isotropic, the interaction energy can be obtained explicitly by substituting eqns (50)–(52) into (55) as

$$\Delta W = -\frac{8}{3}\pi a^3 (\sigma_{33}^0)^2 \frac{W_{3443}}{W_{3333}W_{3443} - W_{3343}^2}, \tag{56}$$

$$\Delta W = -\frac{8}{3}\pi a^3 (D_3^0)^2 \frac{W_{3333}}{W_{3333}W_{3443} - W_{3343}^2}, \tag{57}$$

$$\Delta W = -\frac{8\pi a^3 (\sigma_{31}^0)^2}{3W_{1313}}, \tag{58}$$

where the following definitions have been made:

$$W_{3333} = \frac{a}{\pi}\{2C_{13}^2(\Pi_{1111}+\Pi_{1212})+4C_{13}(C_{33}\Pi_{1313}+e_{33}\Pi_{1413})+C_{33}^2\Pi_{3333}$$
$$+2C_{33}e_{33}\Pi_{3433}+e_{33}^2\Pi_{4433}\},$$

$$W_{3343} = W_{3433} = \frac{a}{\pi}\{2C_{13}(e_{31}\Pi_{1111}+e_{31}\Pi_{1212}+e_{33}\Pi_{1313}-\kappa_{33}\Pi_{1413})+C_{33}(2e_{31}\Pi_{1313}$$
$$+e_{33}\Pi_{3333}-\kappa_{33}\Pi_{3433})+2e_{31}e_{33}\Pi_{1413}+e_{33}^2\Pi_{3433}-e_{33}\kappa_{33}\Pi_{4433}\},$$

$$W_{3443} = \frac{a}{\pi}\{2e_{31}^2(\Pi_{1111}+\Pi_{1212})+4e_{31}(e_{33}\Pi_{1313}-\kappa_{33}\Pi_{1413})+e_{33}^2\Pi_{3333}$$
$$-2e_{33}\kappa_{33}\Pi_{3433}+\kappa_{33}^2\Pi_{4433}\},$$

$$W_{1313} = \frac{a}{\pi}\{C_{44}^2(\Pi_{1133}+2\Pi_{1313}+\Pi_{3311})+2C_{44}e_{15}(\Pi_{1413}+\Pi_{3411})+e_{15}^2\Pi_{4411}\}. \tag{59}$$

The critical stress and electric displacement for a given penny-shaped crack to grow can be determined from the Griffith (1921) fracture criterion

$$\frac{\partial}{\partial a}(\Delta W + 2\pi a^2 \gamma) = 0, \tag{60}$$

where γ denotes the surface energy density of the piezoelectric material. Substituting eqns (56)–(58) into condition (60) leads to

$$\sigma_{33}^c = \sqrt{\frac{\gamma(W_{3333}W_{3443} - W_{3343}^2)}{2aW_{3443}}} \tag{61}$$

for the critical tensile stress,

$$D_3^c = \sqrt{\frac{\gamma(W_{3333}W_{3443} - W_{3343}^2)}{2aW_{3333}}} \tag{62}$$

for the critical electric displacement, and

$$\sigma_{31}^c = \sqrt{\frac{\gamma W_{1313}}{2a}} \tag{63}$$

for the critical shear stress.

When piezoelectric coupling is absent and the material is isotropic, the expressions in eqn (59) can be shown as

$$W_{3333} = \frac{2\pi\mu}{1-\nu}, \quad W_{3343} = W_{3433} = W_{3443} = 0, \quad W_{1313} = \frac{\pi\mu(2-\nu)}{1-\nu}, \tag{64}$$

in which μ and ν are the shear modulus and Poisson's ratio, respectively. On substituting the above equation into eqns (61) and (63), the critical tensile and shear stress, respectively, become

$$\sigma_{33}^{c} = \sqrt{\frac{\pi\mu\gamma}{(1-\nu)a}}, \tag{65}$$

and

$$\sigma_{31}^{c} = \sqrt{\frac{\pi\mu\gamma(2-\nu)}{2(1-\nu)a}} \tag{66}$$

which agree with the results of Sack (1946) and Sneddon (1946).

STRESS INTENSITY FACTORS

If the matrix and inclusion are perfectly bonded together, the adjacent points just inside and just outside inclusion suffer no relative elastic displacement and electric field, and that traction and normal component of the electric displacement across the boundary are continuous. Namely,

$$[U_J] = U_J^{\mathrm{in}} - U_J^{\mathrm{out}} = 0, \tag{67}$$

$$\left[\sum_{iJ}\right] n_i = \left(\sum_{iJ}^{\mathrm{in}} - \sum_{iJ}^{\mathrm{out}}\right) n_i = 0, \tag{68}$$

where $[\cdot]$ denotes the jump in the quantity under consideration at the boundary between the inclusion and the surrounding material, and n_i is the unit outward normal to the surface of the inclusion. The preceding superscripts 'in' and 'out' denote quantities just inside and just outside the inclusion, respectively.

Since the eigenfield Z_{Mn}^{*} is discontinuous on the boundary of the inclusion Ω, the elastic displacement gradient and electric field $U_{j,i}$ are discontinuous at the interface. For a line element $\mathrm{d}x_i$ on the interface, the jump across the interface can be written as

$$[U_{J,i}]\,\mathrm{d}x_i = 0. \tag{69}$$

The above equation is satisfied by choosing

$$[U_{J,i}] = \lambda_J n_i, \tag{70}$$

since $n_i \mathrm{d}x_i = 0$. The unknown magnitude of the jumps, λ_J ($J = 1 \sim 4$), is to be determined.

When the constitutive equation

$$\left[\sum_{iJ}\right] = \hat{C}_{iJAb}([U_{A,b}] - [Z_{Ab}^{*}]), \tag{71}$$

and eqn (70) are used, eqn (68) becomes

$$\hat{C}_{iJAb}n_i n_b \lambda_A = -\hat{C}_{iJAb}Z^*_{Ab}n_i. \quad (72)$$

It should be emphasized that $Z^*_{Ab} = 0$ in the matrix. Now, solving for four unknowns λ_A from the preceding system of simultaneous equations with given n_i and Z^*_{Mn}, we have

$$\lambda_A = \hat{C}_{iJMn}Z^*_{Mn}n_i N_{AJ}(\mathbf{n})/D(\mathbf{n}), \quad (73)$$

where $N_{AJ}(\mathbf{n})$ and $D(\mathbf{n})$ are, respectively, the cofactor and the determinant of matrix $\hat{C}_{pAJq}n_p n_q$. Thus, the jump of stress and electric displacement at the surface of the inclusion is found by substituting eqns (70) and (73) into (71) as

$$\left[\sum_{iJ}\right] = \hat{C}_{iJAb}\{Z^*_{Ab} - \hat{C}_{pQMn}Z^*_{Mn}G_{AQbp}(\mathbf{n})\}, \quad (74)$$

where $G_{AQbp}(\mathbf{n}) = N_{AQ}(\mathbf{n})n_b n_p/D(\mathbf{n})$. The foregoing equation is similar to that obtained by Dunn (1994).

Then, for a given applied electromechanical load Σ^0_{iJ}, the stress and electric displacement immediately outside the inclusion are obtained as

$$\begin{aligned}\overset{\text{out}}{\sum_{iJ}} &= \overset{0}{\sum_{iJ}} + \overset{\text{in}}{\sum_{iJ}} + \left[\sum_{iJ}\right] \\ &= \overset{0}{\sum_{iJ}} + \hat{C}_{iJMn}(S_{MnAb} + I_{MnAb})Z^*_{Ab} - \hat{C}_{iJAb}C^*_{pQMn}Z^*_{Mn}G_{AQbp}(\mathbf{n}). \quad (75)\end{aligned}$$

Once the eqn (75) is evaluated, the stress and electric displacement concentration factors of the inclusion is easily defined in a unified form as

$$F = \overset{\text{out}}{\sum_{iJ}} \Big/ \overset{0}{\sum_{iJ}}. \quad (76)$$

For the crack problem ($C^*_{ijmn} = e^*_{imn} = 0$, and $\kappa^*_{in} \approx 0$), the concentration factor in eqn (76) can be further reduced to

$$F = \hat{C}_{iJAb}\{Z^*_{Ab} - \hat{C}_{pQMn}Z^*_{Mn}G_{AQbp}(\mathbf{n})\} \Big/ \overset{0}{\sum_{iJ}}. \quad (77)$$

In the following, the stress and electric displacement intensity factors corresponding to the three loading cases will be determined.

Simple tension case. $\overset{0}{\sum_{iJ}} = \sigma^0_{33}$.

Combining eqns (50)–(53) and (77), the stress concentration factor for simple tension case is given by

$$F = \frac{(C_{33}L_{3443} - e_{33}L_{3433}) - \hat{C}_{33Ab}(\hat{C}_{pQ33}L_{3443} - \hat{C}_{pQ43}L_{3433})G_{AQbp})(\mathbf{n})}{a_3(L_{3333}L_{3443} - L_{3343}L_{3433})}. \quad (78)$$

Denote $\rho = a_3^2/a$ as the root radius at the penny-shaped crack surface locating on the plane containing the normal vector n_i and the line parallel to the x_3 axis. If expressed in terms of the root radius ρ, eqn (78) can be written as

$$F = \frac{(C_{33}L_{3443} - e_{33}L_{3433}) - \hat{C}_{33Ab}(\hat{C}_{pQ33}L_{3443} - \hat{C}_{pQ43}L_{3433})G_{AQbp}(\mathbf{n})}{(L_{3333}L_{3443} - L_{3443}L_{3433})\sqrt{\rho a}}. \tag{79}$$

The Mode I stress intensity factor can then be determined from the above equation as $\rho \to 0$:

$$K_I = \lim_{\rho \to 0} F\sqrt{\rho}\sigma^0_{33}. \tag{80}$$

For an elastically isotropic material, K_I is reduced to

$$K_I = \frac{2}{\pi}\sigma^0_{33}\sqrt{\pi a} \tag{81}$$

which agrees with Sneddon's (1946) solution.

Electric displacement case. $\overset{0}{\sum_{iJ}} = D^0_3$.

Substituting eqns (50)–(53) into (77), the stress concentration factor in terms of the root radius ρ on $x_1 - x_2$ plane for electric displacement case is written as

$$F = \frac{(e_{33}L_{3333} - C_{33}L_{3343}) - \hat{C}_{34Ab}(\hat{C}_{pQ43}L_{3333} - \hat{C}_{pQ33}L_{3343})G_{AQbp}(\mathbf{n})}{(L_{3333}L_{3443} - L_{3343}L_{3433})\sqrt{\rho a}}. \tag{82}$$

The electric displacement intensity factor for this case is defined by

$$K^D_I = \lim_{\rho \to 0} F\sqrt{\rho}D^0_3. \tag{83}$$

Pure shear case. $\overset{0}{\sum_{iJ}} = \sigma^0_{31}$.

In the same manner as the previous cases, the stress concentration factor and the Mode III type stress intensity factor for pure shear case are respectively defined by

$$F = \frac{C_{31} - \hat{C}_{31Ab}\hat{C}_{pQ31}G_{AQbp}(\mathbf{n})}{L_{1313}\sqrt{\rho a}}, \tag{84}$$

and

$$K_{III} = \lim_{\rho \to 0} F\sqrt{\rho}\sigma^0_{31}. \tag{85}$$

CONCLUDING REMARK

The anisotropic inclusion method has been extended to investigate the problem of flat ellipsoidal inclusions in a transversely isotropic piezoelectric material. By assuming the inclusion is filled with vacuum, and the permittivity of the matrix is much larger than that of the vacuum, the method is further applied to examine the fracture problem of a penny-shaped crack embedded in a piezoelectric material subjected to a set of mechanical and electric loads. The results have indicated that the applied shear stress does not couple with the applied electric displacement, unlike the simple tension case. Furthermore, based on the Griffith fracture criterion, fracture stresses and critical electric displacement of the penny-shaped crack have been obtained in closed forms. The fractures stresses have been shown to reduce those for uncoupled linear elastic fracture mechanics when piezoelectric coupling is absent and the material is isotropic. Finally, the explicit expressions for stress and electric displacement intensity factors have also been given. It is noted that the stress intensity factors defined in this paper are the limiting case of the stress concentration factors defined when a_3 is still finite, multiplied by the root radius and the applied load. As shown in eqn (81), the resulting stress intensity factors are also in coincidence with those obtained in linear elastic fracture mechanics if the material is purely elastic. This confirms that the formulation obtained in the present paper is applicable not only to the thin crack in piezoelectric media, but also to elastic anisotropic flat ellipsoidal crack problems.

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REFERENCES

Deeg, W. F. (1980). The analysis of dislocation, cracks, and inclusion problems in piezoelectric solids, Ph.D. dissertation, Stanford University.

Dunn, M. L. and Taya, M. (1993). Micromechanics predictions of the effective electroelastic moduli of piezoelectric composites. *International Journal of Solids and Structures* **30,** 161–175.

Dunn, M. L. (1994). The effects of crack face boundary conditions on the fracture mechanics of piezoelectric solids. *Engineering Fracture Mechanics* **48,** 25–39.

Eshelby, J. D. (1957). The determination of the elastic field of an ellipsoidal inclusion, and related problems. *Proceedings of the Royal Society* **A421,** 376–396.

Griffith, A. A. (1921). The phenomena of rupture and flow in solid. *Transactions of the Royal Society* **A221,** 163–179.

Huang, J. H. and Yu, J. S. (1994). Electroelastic Eshelby's tensors for an ellipsoidal piezoelectric inclusion. *Composite Engineering* **4,** 1169–1182.

Mura, T. (1987). *Micromechanics of Defects in Solids*, 2nd revised edn, Martinus Nijhoff Publishers, Dordrecht.

Pak, Y. E. (1992). Circular inclusion problem in antiplane piezoelectricity. *International Journal of Solids and Structures* **29,** 2403–2419.

Parton, V. Z. (1976). Fracture mechanics of piezoelectric materials. *Acta Astronautica* **3,** 671–683.

Sack, R. A. (1946). Extension of Griffith's theory of rupture to three dimensions. *Proceedings of the Physics Society* **58,** 729–736.

Sneddon, I. N. (1946). The distribution of stress in the neighborhood of a crack in an elastic solid. *Proceedings of the Royal Society* **A187,** 229–260.

Sosa, H. A. and Pak, Y. E. (1990). Three-dimensional eigenfunction analysis of a crack in a piezoelectric material. *International Journal of Solids and Structures* **26,** 1–15.

Sosa, H. A. (1991). Plane problems in piezoelectric media with defects. *International Journal of Solids and Structures* **28,** 491–505.

Suo, Z., Kuo, C. M., Barnett, D. M. and Willis, J. R. (1992). Fractures mechanics for piezoelectric ceramics. *Journal of Mechanics, Physics and Solids* **40,** 739–756.

Wang, B. (1992). Three-dimensional analysis of an ellipsoidal inclusion in a piezoelectric material. *International Journal of Engineering Science* **29,** 293–308.

Wang, B. (1994). Three-dimensional analysis of a flat elliptical crack in a piezoelectric material. *International Journal of Engineering Science* **30,** 781–791.

Wang, Z. (1994). Penny-shaped crack in transversely isotropic piezoelectric materials. *Acta Mechanica Sinica* **10,** 49–60.

APPENDIX

The coefficients appear in Π_{1133} are listed below:

$$\alpha_0 = a_0d_0(a_{11}+b_{11}),$$

$$\alpha_1 = 2a_0d_0(2d_{11}+f_{11}+h_{11}), \tag{A1}$$

$$\alpha_2 = 2b_0d_0d_{11}-a_{11}b_0f_0-b_0b_{11}f_0+2a_0d_{11}f_0+b_0d_0f_{11}+a_0f_0f_{11}-a_0a_{11}h_0 \\ -a_0b_{11}h_0+b_0d_0h_{11}+a_0f_0h_{11}+6a_0d_0m_{11}+3a_0d_0p_{11}+3a_0d_0q_{11}, \tag{A2}$$

$$\alpha_3 = 2(-a_0b_0h_0-b_0b_{11}h_0-a_0a_{11}m_0-a_0b_{11}m_0+2b_0d_0m_{11}+2a_0f_0m_{11} \\ +b_0d_0p_{11}+a_0f_0p_{11}+b_0d_0q_{11}+a_0f_0q_{11}+4a_0d_0r_{11}), \tag{A3}$$

$$\alpha_4 = -2b_0d_{11}h_0-b_0h_0f_{11}-b_0h_0h_{11}-3a_{11}b_0m_0-3b_0b_{11}m_0-2a_0d_{11}m_0-a_0f_{11}m_0 \\ -a_0h_{11}m_0+2b_0f_0m_{11}+2a_0h_0m_{11}+b_0f_0p_{11}+a_0h_0p_{11}+b_0f_0q_{11}+a_0h_0q_{11}+6b_0d_0r_{11}+6b_0f_0r_{11},$$

$$\alpha_5 = 2(-2b_0d_{11}m_0-b_0f_{11}m_0-b_0h_{11}m_0+2b_0f_0r_{11}+2a_0h_0r_{11}), \tag{A5}$$

$$\alpha_6 = -2b_0m_0m_{11}-b_0m_0p_{11}-b_0m_0q_{11}+2b_0h_0r_{11}+2a_0m_0r_{11}, \tag{A6}$$

where $a_0 \sim m_0$ and $a_{IJ} \sim r_{IJ}$ have been given in the paper of Huang and Yu (1994).